Finite-size correction in a disordered system: A new divergence

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We show that the amplitude of the finite-size correction term for the nth moment of the partition function, for randomly interacting directed polymers, diverges (on the high temperature side) as $(n_c - n)^{-r}$, as a critical moment n_c is approached. The exponent r is independent of temperature but does depend on the effective dimensionality. There is no such divergence on the low temperature side ($n > n_c$).

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The averages of thermodynamic quantities for quenched random systems are often done through the replica trick, which requires an analytic continuation of the integer moments of the partition function $\overline{Z^n}$ to $n \to 0$ (see, e.g., [1]), with the overbar representing the disorder average. It is only recently that several issues related to this continuation could be debated quantitatively because of the results available for the problem of a directed polymer in a random medium [2]. This was not feasible for other models. For example, the replica prediction of the nonexistence of the higher-order (> 3) cumulants of the free energy of the (1+1)-dimensional directed polymer problem has been challenged by direct numerical computation of those [3,4]. In large dimensions, the behaviors of the moments for $n > 1$ and $n < 1$ are found to be different [5]. The possibility of the existence of a singularity in n has also been pointed out [6,7]. All of these raise doubt about the analytic continuation and interchange of the $n \to 0$ with the thermodynamic limit. In complex situations such as the spin glass, even the effective Hamiltonian for the nth moment can admit of many phases that make analytic continuation nontrivial [8].

In this paper we show an even more striking possibility in a directed polymer system. It is a *divergence* of the finite-size effect for the moments of the partition function in the *high temperature phase* as a critical moment is approached. This divergence is distinct from the known singularity, at the thermodynamic transition point, in the finite-size correction to the free energy for disordered systems [11] such as spin glasses [9], the random energy model [10], and a directed polymer in a random medium [6]. In contrast, the divergence we report here is not necessarily at the bulk transition point.

Our model involves two directed polymers that interact on contact. The interaction energy is random with zero mean. The model is similar to those used for depinning transitions of linelike objects [12]. In the continuum version, the random interaction model is known to exhibit a disorder induced transition from a weak disorder (high temperture) to a strong disorder (to be called the RANI phase) phase [13,14]. For a detailed understanding of the phases, we study the model on a lattice using a real-space renormalization group approach. Since a real-space renormalization group can be implemented exactly on hierarchical lattices [15], we consider the problem from the beginning on a hierarchical lattice. As usual, the procedure can be thought of as an approximate real-space renormalization group on a regular lattice.

The lattice, as mentioned, is generated hierarchically by replacing each bond at the μth generation by a "diamond" of b branches (Fig. 1). Two directed polymers start at the bottom of the lattice and meet at the upper end, without any backtracking. The model of interest is the site model where the two polymers interact whenever they meet at a site, with a random interaction energy ϵ. For simplicity, all the directed paths are taken to be identical as far as the choice of ϵ on these is concerned. However, the energies along a particular path are independent and uncorrelated, so that the randomness is only in the longitudinal (special) direction and not in the transverse direction (see Fig. 1). For this paper, the only information needed about randomness is the behavior of the moments of the weight $y = \exp(-\epsilon/T)$ (T being the temperature in units of the Boltzmann constant), which, for a Gaussian distribution of ϵ, behaves as $\overline{y^m} = \overline{y}^{m^2}$. Other distributions have been considered, but will not be elaborated upon here.

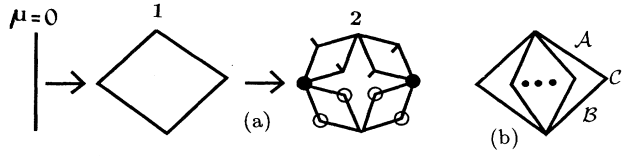

FIG. 1. Construction of a hierarchical lattice (a) for b=2. Three generations (μ=0,1,2) are shown. Identically marked points are taken to have the same interaction energy ϵ, but ϵ is random for points with different markers. (b) A more general motif with $2b$ bonds.

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Let Z_μ be the partition function for a given realization of randomness and let $S_\mu = b^{L_\mu - 1}$ be the number of single-chain configurations, at the μth generation. Here $L_\mu = 2^\mu$ is the length of a directed polymer. We define $\mathcal{Z}_\mu(n) = \overline{Z_\mu^n}/S_\mu^{2n}$, to factor out the free chain entropy, and call $\mathcal{Z}_\mu(n)$ the moments. We establish that, like the random medium problem [7], for a given temperature, there is a critical value $n = n_c(\overline{y})$ below which all moments are in their high temperature phase, in the thermodynamic limit of course. In this limit, $\mathcal{Z}_\mu(n)$ approaches a fixed point value for $n < n_c$, whereas, for $n > n_c$, the moments diverge but with a finite "free energy" density $f_\mu(n) \equiv (nL_\mu)^{-1} \ln \mathcal{Z}_\mu(n)$ (the RANI phase). The approach to the thermodynamic limit can be written generically as

$$g_\mu(n) = g(n) + B_g(n)\, L_\mu^{-\psi_g} + \cdots, \tag{1}$$

where $g(n)$ is the thermodynamic limit ($\mu \to \infty$) and $B_g(n)$ is the amplitude of the finite size correction. We take $g_\mu(n)$ for $n < n_c$ to be $[\mathcal{Z}_\mu(n)]^{1/n}$ and for $n > n_c$ it is $f_\mu(n)$. We will see that the corrections are power laws in L.

One of the main results of the paper is the blowing up of $B_z(n)$ as $(n_c - n)^{-r}$, for $n \to n_c-$, a feature whose existence, as far as we know, has not been recognized before. No such divergence occurs for $n > n_c$. We study the variation of r with the various parameters of the problem and establish its universality for a given distribution. We also characterize the behavior of $f_\mu(n)$ for $n \to n_c+$.

For a given realization of disorder, the partition function can be written as (see Fig. 1)

$$Z_{\mu+1} = bZ_\mu^{(\mathcal{A})} y Z_\mu^{(\mathcal{B})} + b(b-1)S_\mu^4. \tag{2}$$

The first term originates from the configurations that require the two directed polymers to meet at C, while the second term counts the "no encounter" cases. There are no energy costs at the two end points. The moments of the partition function, from Eq. (2), are

$$\mathcal{Z}_{\mu+1}(n) = b^{-n} \sum_{m=0}^{n} P_{nm} \mathcal{Z}_\mu^2(m), \tag{3}$$

where $P_{nm} = \binom{n}{m}(b-1)^{n-m}\overline{y^m}$, with the initial condition $\mathcal{Z}_0(n) = 1$ for all the moments because there is no interaction in the zeroth generation (one single bond).

The behavior of the moments is determined by the stable fixed points of the recursion relation as $\mu \to \infty$. The first moment has no fixed point for $\overline{y} > y_1 \equiv b^2/[4(b-1)]$. For $y_1 \equiv \overline{y}/y_1 < 1$, $\mathcal{Z}_\mu(1)$ reaches a fixed point value $\mathcal{Z}^*(1)$ for large μ. For $y_1 > 1$, $f_\mu(1)$ approaches a definite limit. For higher moments, the stable fixed points in extended spaces can be determined with high precision on a computer by just solving a quadratic equation. It is easy to see that, for the stable fixed points of the first $n-1$ moments, there is again a critical value y_n

$$y_n^{-1} = 4b^{-2n} \sum_{m=0}^{n-1} P_{nm} [\mathcal{Z}^*(m)]^2,$$

so that for $y_n \equiv \overline{y^n}/y_n > 1$ there is no fixed point for the nth moment even though the lower moments do have fixed points. The stable fixed point, if it is real, can be written as

$$\mathcal{Z}^*(n) = b^n (2\overline{y^n})^{-1} \left[1 - (1 - y_n)^{1/2}\right]. \tag{4}$$

The property to be used later is that $2\mathcal{Z}^*(n)\overline{y^n}b^{-n} < 1$.

In our numerical analysis, by iterating the recursion relations for given b and $\overline{y}$, we computed the moments of the partition function with large (70 digit) accuracy using MATHEMATICA. Iterations up to 100 generations are done and to check them, several cases with 300 generations are also considered. For a given $\overline{y}$, there is a critical value n_c so that for $n < n_c$ the moments reach their fixed point value, as shown in the lower inset of Fig. 2. This n_c depends on $\overline{y}$ (i.e., temperature), b, and the distribution. For $n < n_c$, the moments are in the high temperature phase, as per Eq. (4), because for these $y_n > \overline{y^n}$. By

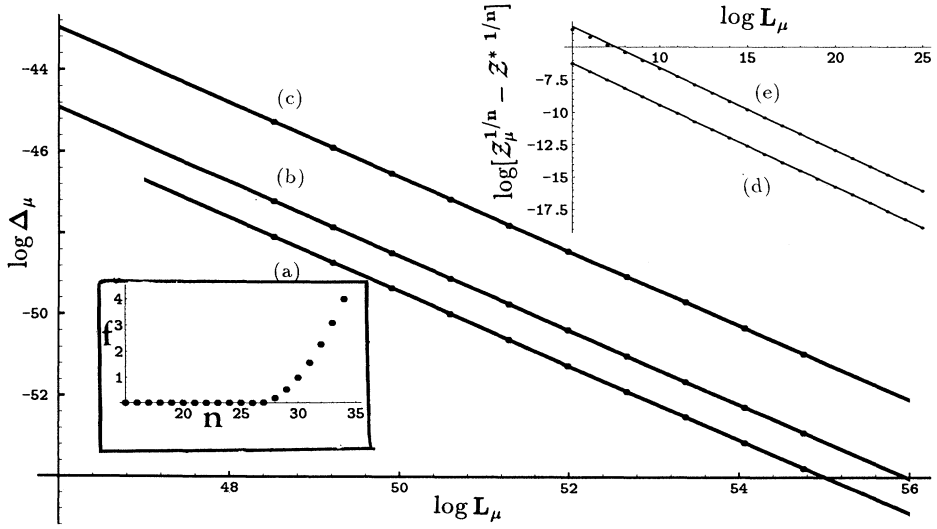

FIG. 2. Plot of $\ln \Delta_\mu(n)$ vs $\ln L_\mu$ for $n = 5$ [curve (a)], $n = 20$ [curve (b)], and $n = 26$ [curve (c)]. With $b = 4$ and $\ln \overline{y} = 0.04$. The straight lines are the fits to these log-log plots. The lower inset shows the plot of $f(n)$ vs n. The transition is at $n_c = 26.6$. The lower moments $n < n_c$ are in the high temperature phase while the higher ones are in the low temperature RANI phase. The upper inset compares Eq. (1) with data points for $n = 5$ [curve (d)] and $n = 26$ [curve (e)] on a logarithmic scale.

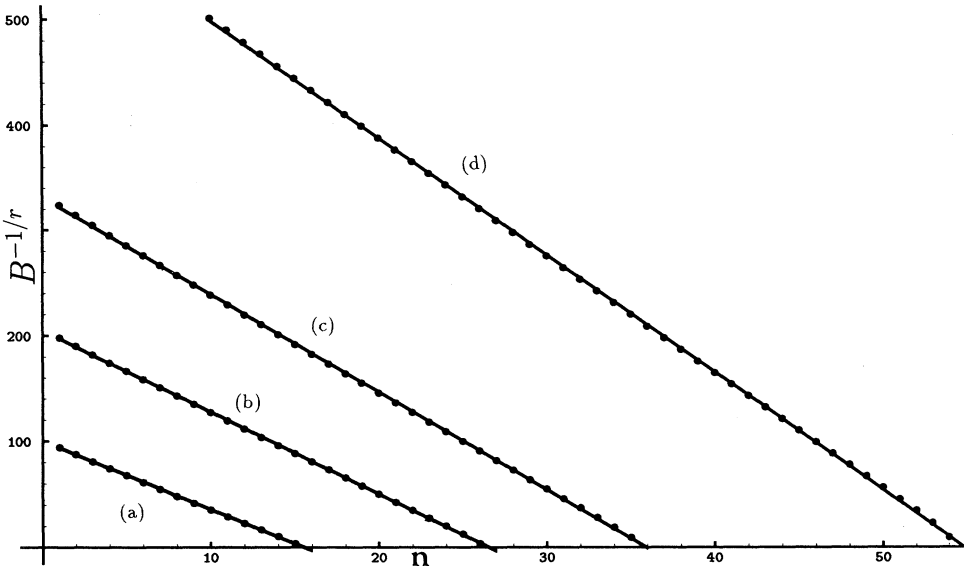

FIG. 3. Plot of $[B(n)]^{-1/r}$ vs n for $b = 4$ and (a) $\log_{10}\overline{y} = 0.065$ and $r = 0.73$, (b) $\log_{10}\overline{y} = 0.04$ and $r = 0.73$, (c) $\log_{10}\overline{y} = 0.03$ and $r = 0.72$, and (d) $\log_{10}\overline{y} = 0.02$ and $r = 0.72$. The straight lines are the best fits through the points.

the same token, for $n > n_c$, the moments are in the low temperature phase.

To analyze the finite-size data (finite generations) for $n < n_c$, we adopt the following numerical procedure. First, construct the differences $\Delta_\mu(n) \equiv \mathcal{Z}_{\mu+1}^{1/n}(n) - \mathcal{Z}_\mu^{1/n}(n)$, from Eq. (1), as

$$\Delta_\mu(n) = B(n)(1 - 2^{-\psi})\, L_\mu^{-\psi}, \tag{5}$$

omitting, for simplicity, the subscript z. Hence a log-log plot would give ψ and $B(n)$ provided μ is sufficiently large. Such an analysis has been done for all the moments for various $\overline{y}$ and b. The sample plots of Fig. 2 clearly show that the exponent ψ is independent of n, however, it depends on $\overline{y}$. (The upper inset of Fig. 2 shows a few sample plots for $[\mathcal{Z}_\mu(n)]^{1/n} - \mathcal{Z}^*(n)^{1/n}$ vs μ against $B(n)L_\mu^{-\psi}$ with the estimated values.) Figure 3 shows the growth of the amplitude with the moment indicating a divergence as the critical n_c is approached from below. The location of n_c and the exponent r, as defined after Eq. (1), can be determined by choosing r such that $B^{-1/r}(n)$ is a straight line with n. The intercept gives n_c.

A similar analysis is done for $f_\mu(n)$ for $n > n_c$. The exponent $\psi = 1$ and there is no divergence of the amplitude near n_c. The finite size correction, from Eq. (3), is $(\ln \overline{y^n} - n \ln b)L^{-1}$. In this case, the free energy, $f(n)$ in the thermodynamic limit vanishes in a singular fashion as n_c is approached. Assuming $f(n) \sim (n - n_c)^\sigma$, σ and n_c can be estimated by linearizing the $f^{1/\sigma}(n)$ vs n plot.

The estimates of n_c from the two sides are consistent with each other. For $n < n_c$ (see Fig. 3), fits over a wide range in n give the exponent r as 0.71 ± 0.02 and this value is independent of temperature. We have checked this insensitivity to temperature up to $\ln \overline{y} = 0.005$ or $n_c = 223.53$. There is definitely a curvature near n_c and

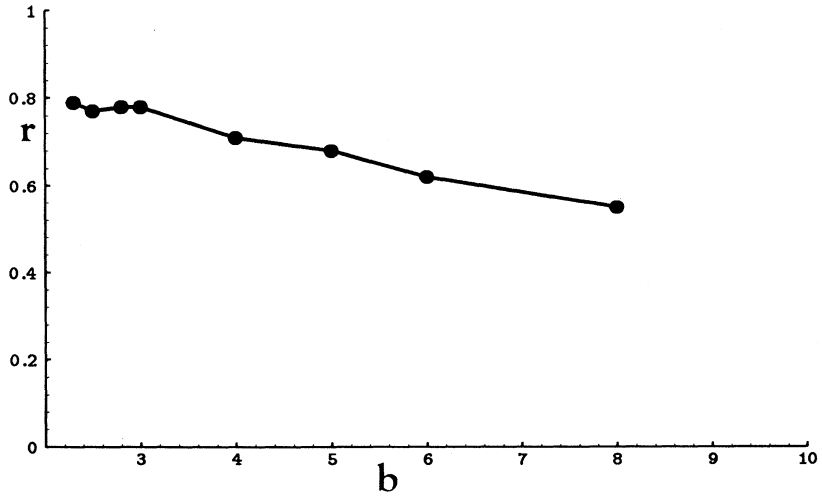

FIG. 4. Dependence of the exponent r on b. The line through the points is just a guide to the eye.

if only the last three or four points are taken, a lower value for r is obtained. However, we believe that this or any other more elaborate procedure to estimate r is not warranted because the last few points are still far from n_c. Therefore, we tend to believe the obtained value of r to be an upper limit such as a mean field estimate. The free energy exponent σ is found to be temperature dependent, i.e., nonuniversal. For example, for $b = 4$, σ changes from 1.78 ± 0.03 at $\ln \bar{y} = 0.04$ to $\sigma = 1.90 \pm 0.05$ at $\ln \bar{y} = 0.08$. What really happens at n_c eludes us because it is not possible to hit an integer n_c numerically. It is plausible that $n \to n_c\pm$ and $n = n_c$ are to be treated separately.

There is a strong dependence of the exponents on b and the distribution. For the Gaussian distribution, the variation of r with b is shown in Fig. 4. This dependence might be expected because as b is changed, the effective dimensionality $2b = 2^d$ changes. Interestingly enough, the exponent reaches a saturation as $b \to 2$, the minimum value of b for a transition.

To rationalize the results, we now analyze the *linearized renormalization group transformation.* For a given $\bar{y}$, we linearize the recursion relation Eq. (3) up to the moment for which the stable fixed point is reached (i.e., $n < n_c$). Let us start with the situation where the moments are close to the fixed point, and define a column vector $\mathbf{z}$ of size n_0, the integer part of n_c, with $z(n) = \mathcal{Z}_\mu(n) - \mathcal{Z}^*(n)$ as the nth element. The transformation matrix $\mathbf{R}$ that takes the vector to a new one $\mathbf{z}' = \mathbf{R}\mathbf{z}$ is lower triangular with elements $2b^{-n}P_{nm}\mathcal{Z}^*(m)$ for $m \leq n$ and zero otherwise. $\mathbf{R}$ is of size $n_0 \times n_0$.

The eigenvalues of this matrix $\mathbf{R}$ are just the diagonal elements $\lambda_n = 2\mathcal{Z}^*(n)\overline{y^n}b^{-n}$ of which, in all cases, we find λ_1 to be the largest. By Eq. (4), $\lambda_n < 1$ for all $n < n_c$. (See Fig. 5 for the spectrum for a particular case.) If $\hat{\mathbf{e}}_n$ is the nth eigenvector, then, after μ iterations, $\mathbf{z}^{(\mu)} = \sum_k A_k \lambda_k^\mu \hat{\mathbf{e}}_k$, where A_k is the projection of the starting vector along $\hat{\mathbf{e}}_k$. For large μ, it follows that the convergence to the fixed point for the nth moment is as $\mathcal{B}_n L_n^{-\psi}$, with

$$\psi = -\log_2 \lambda_1 = -\log_2 \left[1 - (1 - y_1)^{1/2}\right] \tag{6}$$

for all n, and amplitude $\mathcal{B}_n = A_1 \mathbf{e}_{1,n}$, where $\mathbf{e}_{1,n}$ is the nth element of $\hat{\mathbf{e}}_1$.

There is a universality in the exponent ψ when considered as a function of $y_1 = \bar{y}/y_1$. The numerical values for various temperatures b and distributions can be made to collapse on this curve. Incidentally, this is the same exponent that a pure system would have, with attractive interaction, if expressed as y/y_c, where y_c, is the binding-unbinding transition.

The ratio K_n of $n[\mathcal{Z}^*(n)]^{(n-1)/n} B_n/\mathbf{e}_{1,n}$ is plotted against n in Fig. 5 for the same situation. This shows the proportionality of the two. We have ensured that this proportionality is maintained for all the cases. Hence, we infer that the divergence of the finite-size amplitude is really a consequence of the divergence of the elements of the eigenvector for the largest eigenvalue.

How could the amplitude diverge? The eigenvalues are shown in Fig. 5 for one temperature. In all cases studied, we find the rise of the last few eigenvalues approaching from below the first (largest) eigenvalue (λ_1). This leads to an increase in the components of the first eigenvector $\hat{\mathbf{e}}$, whose ith component $\mathbf{e}_{1,i}$ is of the form

$$(\lambda_1 - \lambda_i)^{-1} \left(1 + \sum_{j=2}^{i-1} \frac{R_{ij}R_{j1}}{\Lambda_j} + \sum_{j,k} \frac{R_{ij}R_{jk}R_{k1}}{\Lambda_j \Lambda_k} + \cdots\right),$$

where $\Lambda_p = \lambda_1 - \lambda_p$. Therefore, we make the following hypothesis. In an analytic continuation in n, the largest eigenvalue of the transformation operator (not necessarily diagonal anymore) will be degenerate; the eigenvalues of the modes at the end, near n_c, will rise and merge with λ_1. It is this degeneracy that leads to the singular behavior and produces a nontrivial exponent r.

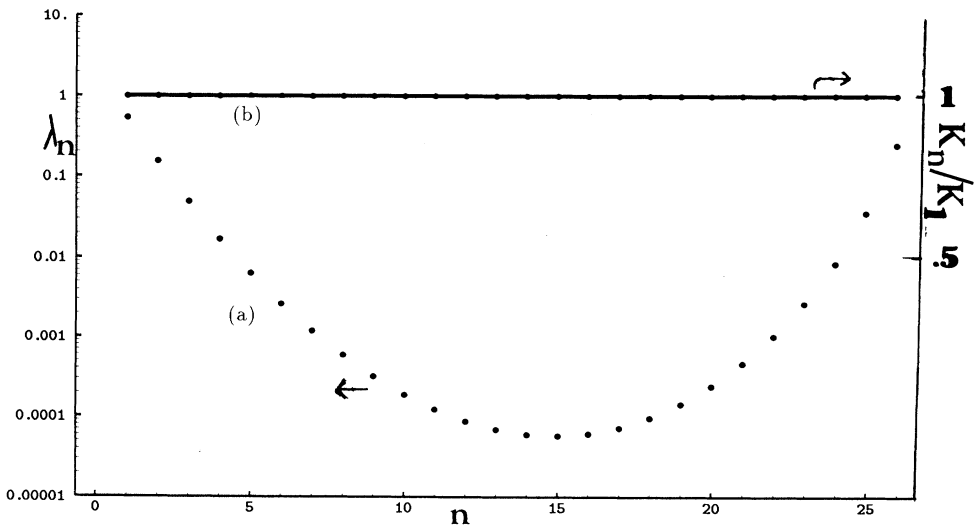

FIG. 5. (a) The 26 eigenvalues of $\mathbf{R}$ for $\ln \bar{y} = 0.04$ for which $n_c = 26.6$. Note the logarithmic scale along the y axis. (b) The horizontal straight line is the ratio K_n/K_1. This plot is on a linear scale, shown on the right.

It is interesting to compare this situation with the problem of a directed polymer in a random medium [6,7,17] and the similar randomly interacting directed polymers but now interacting on the bonds [18]. In these cases, unlike Eq. (2), the temperature does not enter the recursion relations explicitly but only through the initial condition. For this reason, linearizing around the fixed point, with the associated eigenvalues and vectors, will not produce any strong n dependence and hence no divergence in the finite size correction. In fact, an attempt (not serendipity) to formulate a problem where the renormalization group transformations would contain all the important information led us to this particular site version of the random interaction model.

In the context of the random medium directed polymer problem, in 1+1 dimensions, a scaling form has been proposed for the moments $\ln \overline{Z^n} = nLf + g(nL^\omega)$, where ω is the free energy fluctuation exponent and f is the thermodynamic free energy [7]. In this case, the whole phase is the low temperature phase. In our case, for $b > 2$, there is a phase transition that is reflected through a nonzero n_c. The power law growth of the amplitude suggests a different scaling form for $n < n_c$, namely, $\ln \overline{Z^n} = nLf + g[(n_c - n)L^\Omega]$, where $\Omega = \psi/r$. It is tempting to speculate that this Ω is also the free energy fluctuation exponent. The question of scaling on the other side, however, remains an open question.

To summarize, we have shown the existence of a diverging finite-size effect with a weakly universal exponent, at a critical moment from the high temperature side, a scar left by the disorder. The thermodynamic limit on the high temperature side is identical to the pure system. The growth of the amplitude observed numerically can be understood through a renormalization group argument. However, what controls these exponents remains a puzzling issue. A theory to understand the universality (or its absence) of the exponent r is lacking. The full significance of the divergence is yet to be elucidated, but it cautions that an analytic continuation in n has to be done with proper care.

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